CHAPTER 8

Rolling Friction

72. Forces of Rolling Friction. Forces of Sliding Friction in Rolling Motion of a Cylinder

In physics and engineering an important role is played by forces appearing when a wheel (a cylinder) is rolling on a plane. Generally, these forces are called friction forces. To describe the phenomenon in more detail we distinguish between the three types of friction forces which are due to *rolling*, *sliding*, and *adhesion* of wheels. Forces of rolling friction, like friction forces in a translatory motion, always exist and always decelerate the motion. Forces of sliding friction and adhesive forces appearing in a rolling motion can both accelerate and decelerate the rolling body. Like forces of friction of rest, adhesive forces must not necessarily be connected with transformation of mechanical energy into heat.

Let a homogeneous cylinder roll with sliding down inclined plane. Then, generally speaking, all the three types of friction forces are present. In the absence of sliding only forces of rolling friction and adhesive forces act; in many cases the force of rolling friction is very small and then only the adhesion should be taken into account (which was done in Sec. 58). When a rolling motion is accompanied by the action of adhesive forces alone, that is when the point of contact of the cylinder and the plane does not slide relative to the plane, there are no losses of mechanical energy.

Let a cylinder roll on a horizontal plane without rolling friction so that the axis of the cylinder is perpendicular to the velocity of motion. If the cylinder rolls without sliding and its motion is uniform then obviously there is no tangential force of interaction between the cylinder and the plane (it is assumed of course that there is no air resistance).

Now imagine that we take a cylinder rotating about its axis with angular velocity ω_0, put it on a horizontal plane and let it move freely the moment it touches the plane (Fig. 203). Then the cylinder rolls on the plane and the observation shows that at the beginning of its motion there is an acceleration along the direction of motion. This means that the cylinder is acted upon by an external force. This force is exerted by the horizontal plane and the magnitude of the force is equal to the mass of the cylinder times the acceleration of the points lying on the axis of the cylinder. The initial value of the linear velocity of the cylinder is equal to zero and the linear velocity increases as the motion of the cylinder starts whence it follows that during that period of motion sliding takes place. The force of sliding friction thus accelerates the linear velocity of the axis of the cylinder and decelerates its rotation about the axis. Hence, the velocity v of the translatory motion increases while the angular velocity ω decreases. The velocity v_{sl} of the sliding of the points of the cylinder adjoin-

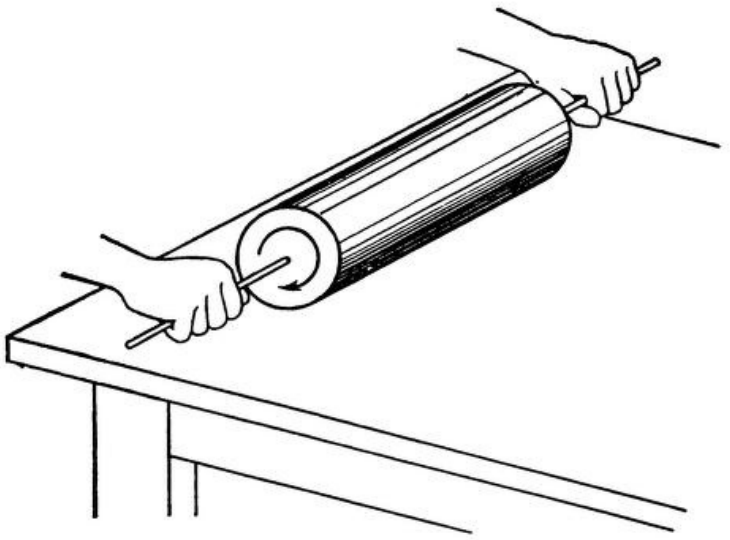

FIG. 203

ing the plane decreases in its modulus because $\omega R > v$ and

$$v_{sl} = \omega R - v$$

where R is the radius of the cylinder. The moment the velocity v_{sl} turns into zero the force of sliding friction stops acting and, beginning with that moment, the motion becomes uniform.

The motion of the cylinder can be investigated quantitatively if we assume that the force of sliding friction in the phenomenon approximately obeys the

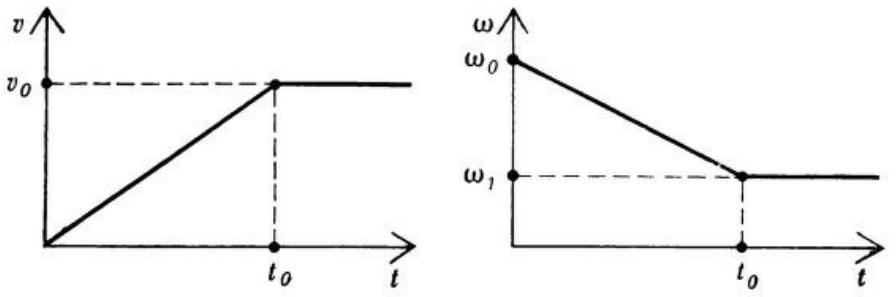

FIG. 204

Coulomb law (see Sec. 42) and is equal to $f_0 = \mu N$ where N is the weight of the cylinder. The moment the cylinder is put on the plane and allowed to move the accelerating force is equal to f_0 and it retains the same value during the motion. Therefore, the linear acceleration of the cylinder is determined by the formula

$$a = \frac{f_0}{m} \tag{72.1}$$

where m is the mass of the cylinder. The cylinder simultaneously possesses a negative angular acceleration $\beta = -|\beta|$ with

$$|\beta| = \frac{f_0 R}{I} \tag{72.2}$$

where I is the moment of inertia of the cylinder with respect to its symmetry axis. Consequently, the velocity of the axis of the cylinder increases according to the linear law

$$v = at = \frac{f_0}{m} t \tag{72.3}$$

while the angular velocity decreases linearly:

$$\omega = \omega_0 - |\beta| t = \omega_0 - \frac{f_0 R}{I} t \tag{72.4}$$

where ω_0 is the initial angular velocity of the cylinder at initial instant $t = 0$ when the rotating cylinder is put on the plane.

It is evident that there is an instant $t = t_0$ ($t_0 > 0$) at which the linear velocity v becomes equal to ωR. At that instant the sliding disappears and, consequently, the force of sliding friction stops acting on the cylinder. That is why after the time instant $t = t_0$ the cylinder rolls without sliding, its axis moving uniformly with velocity

$$v_0 = \frac{\omega_0 R}{1 + \dfrac{mR^2}{I}} \tag{72.5}$$

Formula (72.5) can readily be obtained by eliminating t from expressions (72.3) and (72.4) and putting $\omega R = v_0$. The magnitude t_0 is found from formulas (72.3) and (72.5):

$$t_0 = \frac{mv_0}{f_0} = \frac{m\omega_0 R}{f_0 \left(1 + \dfrac{mR^2}{I}\right)} \tag{72.6}$$

The time diagrams for the linear and angular velocities are shown in Fig. 204. The friction force f_0 acts only from $t = 0$ to $t = t_0$ when sliding takes place. For $t > t_0$ the cylinder rolls uniformly without sliding.

It should be noted that in the phenomenon we have discussed a part of the initial kinetic energy of the rotating cylinder is transformed into heat. Let the reader show that this part of the initial energy is equal to

$$\frac{mR^2 I}{(I + mR^2)^2}$$

We stress once again that at the initial time $t = 0$ there must appear sliding. The uniform motion of the cylinder can only start without sliding at the initial time when there is an impact of the cylinder against the plane at the initial instant, the force of the impact being infinite (more precisely, this force must be sufficiently large). Such an impact can take place when the surface of the plane has some irregularities. After the impact the points of the cylinder contacting the plane stop moving (relative to the plane) under the action of the impulse $P = \int f\,dt$ of the force f exerted by the plane on the cylinder at the points of contact. The impulse P of the force f gives the cylinder a momentum of magnitude $mv = P$ and, besides, decelerates the rotation of the cylinder. The corresponding decrement $\Delta\omega$ of the angular velocity ω can be found from the condition $I\Delta\omega = RP$. After the impact the cylinder is in a pure rolling motion on the plane and $v = (\omega - \Delta\omega) R$. On eliminating P and $\Delta\omega$ we find

$$v = \frac{\omega_0 R}{1 + \dfrac{mR^2}{I}}$$

When the velocity of sliding v_{sl} is directed backwards (i.e. $\omega R > v$) the force of sliding friction is directed forwards; when the velocity v_{sl} is directed forwards the force of sliding friction is directed backwards ($\omega R < v$). In the latter case the friction force decelerates the motion of the cylinder.

Let us make the cylinder roll along the plane with an initial velocity v_0 so that it does not rotate at the initial instant. Then the force of sliding friction decelerates the translatory motion. The velocity v of the translatory motion decreases and the angular velocity ω of the rotational motion increases until ωR becomes equal to v after which a uniform motion without sliding starts.

In a rolling motion without sliding there may appear adhesive forces. In the examples we have discussed there are no such forces only because the cylinder is not acted upon by horizontal external forces. When a cylinder rolling along a plane without sliding is under the action of an external force the corresponding force of adhesion appears. Consequently, adhesive forces, like forces of friction of rest, are specified by the magnitude of the external forces acting on the moving body.

73. Adhesion of Wheels

When an automobile or a locomotive moves and its wheels roll without sliding an important role is played by the forces produced by *adhesion of wheels*. In a rolling without sliding the contacting surfaces of the wheels and of the road or rails do not move relative to each other and there may appear an adhesive force. As in the case of friction of rest, this force does not exceed a definite maximum value f_0. The magnitude f_0 is proportional to the force N with which the wheel presses the supporting surface. The dependence of the adhesive force on the force N can be of a more complex character for a moving automobile because in this case the pressure results in a deformation of the automobile tyres.

Let us consider the scheme of the motion of a self-propelling vehicle (or an automobile) shown in Fig. 205*a*. The driving wheel is under the action of a rotational moment of magnitude $M = f \cdot b$ (Fig. 205*b*) produced by the engine and transmitted to the wheel through the body of the vehicle. The body of the vehicle is in its turn acted upon by a moment of magnitude $M' = f' \cdot b$ directed opposite to M and, by the third law of dynamics, $f = f'$. Since the wheel rolls without sliding the forces applied to it determine its acceleration in accordance with the equalities

$$M - f_{ad}R = I\beta, \quad f_{ad} - f_1 = ma \quad \text{and} \quad a = \beta R \tag{73.1}$$

where f_1 is the force exerted by the body of the vehicle on the axis of the wheel (see Fig. 205*b*), m is the mass of the wheel, f_{ad} is the adhesive force applied to the wheel, I is the moment of inertia of the wheel, and R is its radius. These equations make it possible to establish the relationship between the force f_1 and the moment M:

$$f_1 = \frac{M}{R} - \left(m + \frac{I}{R^2}\right) a \tag{73.2}$$

The force applied to the body of the vehicle is equal to f_1 in the magnitude and opposite in the direction. Hence, f_1 is the magnitude of the driving force producing the motion of the vehicle. It is less than the adhesive force f_{ad} by the pro-

duct ma. When the motion is uniform, that is $a = 0$, formula (73.1) results in

$$f_1 = f_{ad} = \frac{M}{R} \tag{73.3}$$

Consequently, for a uniform motion the driving force f_1 is equal to the adhesive force f_{ad} and its magnitude depends solely on the moment M produced by the engine and on the radius R of the wheel. Formulas (73.1), (73.2) and (73.3) have been derived under the assumption that $f_{ad} < f_0$, which means that there is

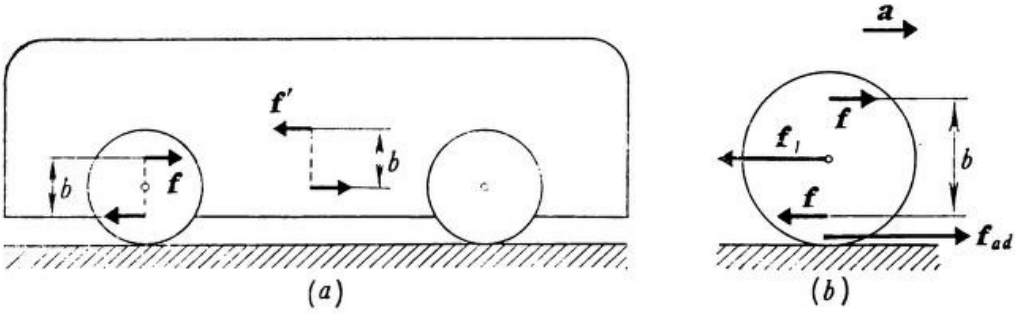

FIG. 205

no sliding of the wheel on the supporting surface. If otherwise, the relation $\omega R = v$ does not apply and, consequently, the equality $\beta R = a$ does not hold either. The external force applied to the driving wheel cannot exceed f_0. When the moment M produced by the engine increases so that the force exceeds that value f_0 sliding sets in. In this case the first two equalities (73.1) remain valid while the last one (i.e. $a = \beta R$) no longer holds; for the case of sliding we have $a < \beta R$. For instance, when $a = 0$ and $\beta > 0$ a part of the rotational moment is expended on the increase of the angular velocity of the wheel but not on the increase of the driving force.

When an automobile starts moving on a slippery road with a sharp increase of the rotational moment M produced by the engine the *slipping* of the wheels may occur. In this case the wheels rotate rapidly and slip while the driving force is small and the automobile does not move or starts moving very slowly. The sharp decrease of the driving force is accounted for by the fact that the magnitude of the force of sliding friction decreases as the velocity of sliding increases. That is why in such cases the moment M should be increased gradually as the automobile starts moving.

In all these considerations we have supposed that the rotational moment M is directed so that it gives the automobile a forward acceleration. When the moment produced by the engine changes sign during the motion the above equations remain valid with the only distinction that the signs of the acceleration a and of the driving force f_1 change. It is evident that in this case the force f_1 is directed backwards and decelerates motion of the automobile.

74. Braking and Skidding

When the brakes of an automobile are applied, the brake shoes (Fig. 206) exert friction forces f' acting on the wheels. These forces produce braking couples with moments M_{br} acting on the wheels and decelerating the automobile.

The moment M_{br} satisfies equations analogous to (73.2) and involving the decelerative force f_d (provided that there is no sliding of the wheels). As has been said, when the automobile starts the rotational moment produced by the engine should be increased gradually in order to avoid the slipping of the wheels and the decrease of the driving force. Similarly, when the brakes are applied, the moment M_{br} of the braking couple must not be increased sharply.

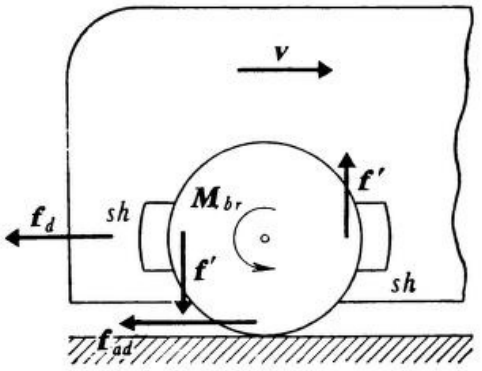

FIG. 206

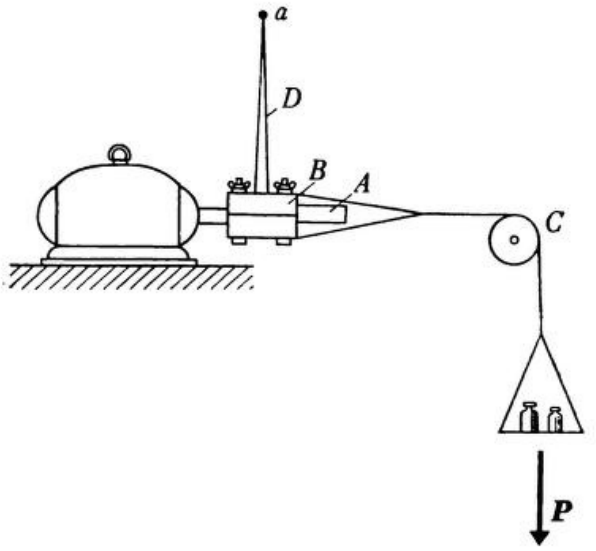

FIG. 207

The sliding of the wheels of an automobile which arises when the brakes are applied may be dangerous because of the phenomenon of *skidding* which leads to the loss of control. Skidding often occurs when the brakes of an automobile are applied on a slippery road. This can also be explained by a specific dependence of the forces of sliding friction on the velocity and the direction of the motion.

When a body slides on a horizontal plane in the direction of its velocity v it turns out that even a very small force f acting perpendicularly to v may produce a considerable displacement of the body in the direction of f. The matter is that a body sliding in a certain direction can produce only a very small friction force as it is displaced in the perpendicular direction. This phenomenon is due to the absence of adhesion between the surfaces of the body and of the plane because the body moves relative to the plane. This effect can be visualized in an experiment suggested by S. E. Khaikin. The scheme of this experiment is shown in Fig. 207.

In the figure we see an electric motor with a clutch B on its axle A. A load of weight P is suspended from a thread which is passed around the pulley C and is attached to the clutch. The force of weight of the load stretches the thread and tends to move the clutch B away from the axle. The clutch consists of two halves pressed together with the aid of screws. On regulating properly the tightening of the screws we can determine the weight P of the load which can remove the clutch from the axle of the electric motor when the axle does not rotate. The clutch is prevented from rotating together with the axle by the lever D attached to the clutch; the other end of the lever is fixed at the point a with the aid of another thread stretched perpendicularly to the plane of the figure. Then the motor is started and the axle rotates; it turns out that the clutch moves along

the axle as the latter rotates under the action of a considerably smaller force (approximately of magnitude of 0.1 P and even less).

This phenomenon of the decrease of the force of sliding friction can be explained rather simply if we take into account that the force of sliding friction is always directed opposite to the velocity of sliding. When the clutch moves along the axle with a velocity v_1 the resultant velocity of the sliding of the particles on the surface of the axle with respect to the clutch is directed at an angle of a small magnitude α to the plane perpendicular to the axle (see Fig. 208). Consequently, the friction force f_0 is directed at the angle α to that plane. Only a small component f_1 of the force f_0 acts in the direction of the axle:

$$f_1 = f_0 \tan \alpha \approx f_0 \alpha \approx f_0 \frac{v_1}{v} \ll f_0 \qquad (74.1)$$

because usually we have $v_1 \ll v$.

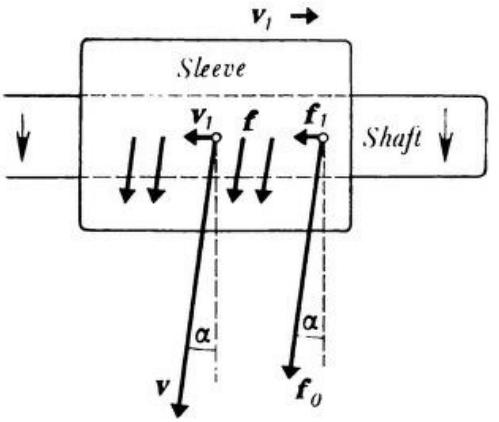

FIG. 208

We now can explain the skidding of an automobile on a slippery road when the brakes are applied sharply. The brake shoes stop the rotation of the wheels and the automobile starts sliding forwards by inertia. A small irregularity on the road or a slight difference in the friction forces acting on different wheels may produce a side force or a moment of forces with respect to the vertical or a lateral displacement of the automobile. For, there are no friction forces (or they are too small) that can prevent the wheels of the automobile from a motion perpendicular to the direction in which the automobile moved before. This is of course a simplified explanation, which however describes correctly the main features of the phenomenon.

75. Rolling Friction

In the foregoing sections we did not discuss the forces of rolling friction. We only dealt with adhesive forces in a rolling motion (analogous to the force of friction of rest) and with forces appearing when a cylinder rolls on a plane with sliding (these forces are analogous to the force of sliding friction between plane surfaces). An important distinction between them is that an adhesive force performs no work (its action is not accompanied by a transformation of mechanical energy into heat) while the force of sliding friction always performs some work connected with energy transformation from the mechanical form into heat.

When a cylinder rolls on a plane without sliding there are always energy losses, that is some amount of mechanical energy goes into heat. The velocity of such a cylinder rolling on a smooth horizontal plane without sliding gradually decreases and finally the cylinder stops. Since the forces of air resistance, which are always present, are very small we conclude that there must be some other force decelerating the cylinder. This is the force of rolling friction whose magnitude depends on the properties of the materials of the cylinder and of the plane.

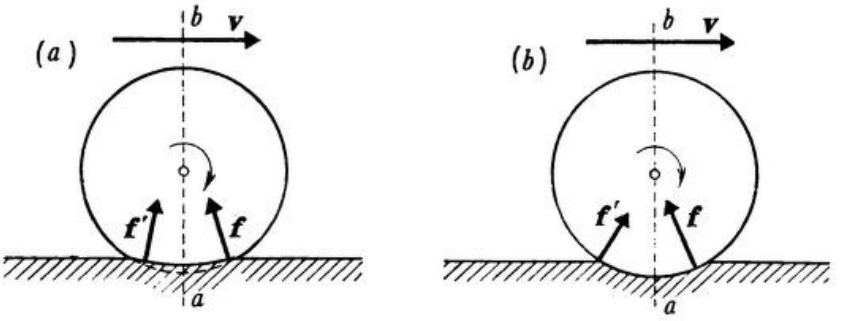

FIG. 209

When the cylinder rolls the plane undergoes a deformation (see Fig. 209*a*) produced by the action of the force pressing the cylinder to the plane. If these deformations are perfectly elastic the interaction forces between the cylinder and the plane must be symmetric with respect to the vertical plane *ab* passing through the axis of the cylinder. To each such force $\boldsymbol{f}$ there corresponds a force $\boldsymbol{f}'$ of the same magnitude which is symmetric to $\boldsymbol{f}$ and whose point of application is symmetric to that of the force $\boldsymbol{f}$. (Here we neglect the forces of sliding friction appearing because of the relative motion of the cylinder and the plane.*)

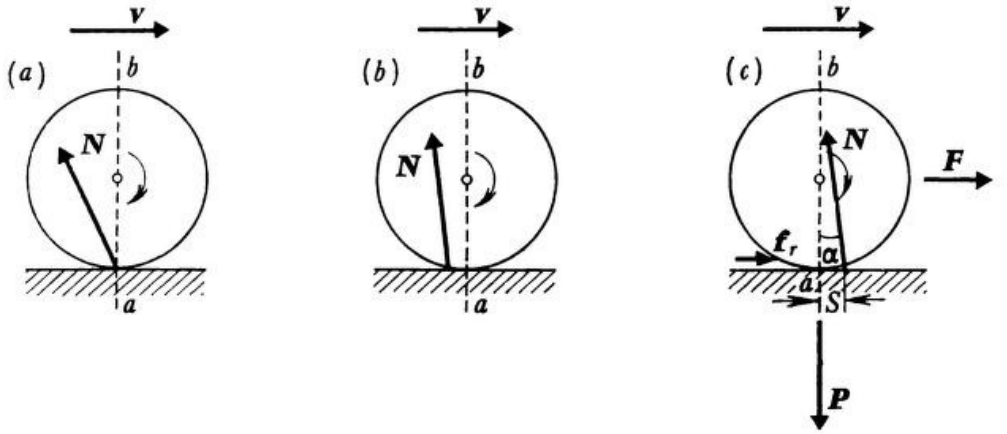

FIG. 210

Because of the symmetry, the resultant of all the forces of the elastic deformation of the supporting surface must be vertical and the sum of the moments of the deformation forces with respect to the axis of the cylinder is equal to zero. Therefore, the forces of elastic deformations of the cylinder and the plane cannot affect the velocity of the rolling motion, that is the motion is such as if there were no deformations at all. This means that no forces of rolling friction arise in such a case.

Thus to explain the appearance of the forces of rolling friction we should assume that the deformation of the cylinder and of the plane is non-elastic, which in fact is always the case.

* Generally speaking, the contacting areas of the cylinder and of the plane always move slightly relative to each other so that the force of sliding friction, however small, inevitably appears.

For the sake of simplicity we assume that the cylinder is non-deformable while the surface of the plane undergoes non-elastic deformation and has some residual deformation. (For the ultimate result of our investigation it is unimportant which of the bodies is in fact deformed, that is the cylinder or the plane or both.) For the model we have chosen it is quite clear that the forces of the action of the plane upon the cylinder are no longer symmetric with respect to the plane *ab* (see Fig. 209*b*): the force f must be of greater magnitude than the force f' applied to the symmetric area lying behind the plane *ab* (relative to the direction of the motion). The resultant force of these deformation forces must therefore have a non-zero horizontal component directed backwards and the sum of the moments of these forces with respect to the axis of the cylinder is also different from zero. The direction of the resultant moment is opposite to that of the direction of the rotation of the cylinder.

An exhaustive theory of rolling friction has not yet been elaborated because the laws describing the forces resulting from variable complex non-elastic deformations have not been investigated thoroughly. That is why we cannot present the exact calculation of forces of rolling friction.

However, taking into account that a cylinder rolling on a horizontal plane without sliding is gradually decelerated, we can draw some qualitative and quantitative conclusions concerning the direction and the magnitude of the friction force exerted on the rolling cylinder by the horizontal plane under the assumption that the area of contact is very small in comparison with the radius of the cylinder.

Let us suppose that the air resistance is negligibly small and can be neglected so that the deceleration of the cylinder can be thought of as being due only to the forces of rolling friction. The cylinder has a negative linear acceleration a and a negative angular acceleration β. These quantities are connected by the condition $a = \beta R$ expressing the absence of sliding. First of all we note that the resultant of all forces acting on the cylinder departs from the vertical in the backward direction (relative to the direction of motion) since the linear acceleration of the cylinder is negative. Now let us determine the position of that resultant force relative to the centre of the cylinder.

The point of application of the resultant force can neither be in the vertical plane *ab* passing through the centre nor behind it, that is the configurations shown in Fig. 210*a* and *b* are impossible. For, if otherwise, the resultant force would give the cylinder a positive acceleration. Consequently, the only possibility is the one shown in Fig. 210*c*: the point of application of the resultant force $\boldsymbol{N}$ must be before the cylinder and the line of action of the force $\boldsymbol{N}$ must intersect the vertical passing through the centre at a point lying above the centre of the cylinder. Indeed, if otherwise, this force would produce a positive acceleration of the cylinder. Thus, the force with which the plane acts upon the cylinder rolling on it without sliding with a deceleration produced by the forces of rolling friction must be directed and applied as shown in Fig. 210*c*.

The horizontal component of the force $\boldsymbol{N}$ is nothing but the force of rolling friction f_r. Since the distance s between the point of application of the force $\boldsymbol{N}$ and the point of contact b is practically very small in comparison with the radius R of the cylinder (this means that the angle of inclination α is very small) the magnitude N of the force $\boldsymbol{N}$ is approximately equal to that of the force pressing the cylinder to the plane, that is to the force of weight of the cylinder $\boldsymbol{P}$ in the case under consideration.

The relationship between the force of rolling friction and other parameters is found experimentally. In principle, this can be performed as follows.

To the axis of the cylinder rolling along the horizontal plane (Fig. 210*c*) a constant force $\boldsymbol{F}$ is applied in the direction of motion and the magnitude F of this horizontal force is chosen so that the motion becomes uniform. Then the magnitude of the force $\boldsymbol{F}$ is obviously equal to that of the force of rolling friction f_r (the air resistance is neglected). The rotation of the cylinder being uniform and its angular acceleration being equal to zero, the force $\boldsymbol{N}$ must pass through the axis of the cylinder. The other two forces, that is the force of weight $\boldsymbol{P}$ and the external force $\boldsymbol{F}$ pass through the axis of the cylinder according to the conditions of the experiment. Consequently,

$$\begin{aligned} P &= N\cos\alpha \text{ and} \\ F &= N\sin\alpha = f_r \end{aligned} \tag{75.1}$$

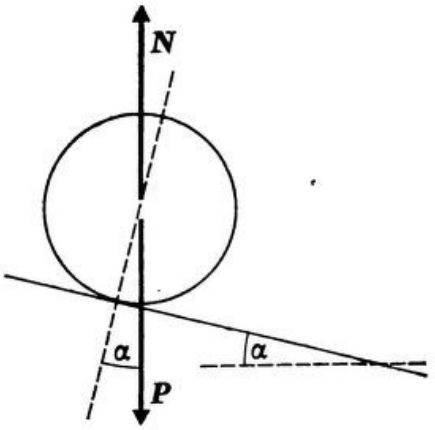

FIG. 211

The angle α is practically very small and relations (75.1) can therefore be rewritten approximately as

$$P \approx N \text{ and } f_r \approx N\alpha \approx P\frac{s}{R} \tag{75.2}$$

There are tables in which the experimental values of the quantity s are compiled. Instead of the expression

$$f_r \approx P\frac{s}{R} \tag{75.3}$$

of the force of rolling friction it is more convenient to write the formula for the moment of the friction force with respect to the axis of the rolling body:

$$f_r R \approx Ps \tag{75.4}$$

Thus, the moment of the force of rolling friction is equal to the product of the force of normal pressure P by the quantity s, which is referred to as the *coefficient of rolling friction.*

Experiments show that, within some limits, for steel, other metals and hard wood the values of s are practically independent of the velocity of the rolling motion and of the radius of the cylinder although, in principle, it is clear that such a dependence, however slight, must exist. Thus, practically the coefficient s depends solely on the materials of the cylinder and of the plane.

The coefficient s can also be determined in experiments with a cylinder rolling down inclined plane. Suppose that the angle of inclination α of the plane (see Fig. 211) is chosen so that the cylinder rolls down the plane uniformly without sliding. In this case the force with which the plane acts upon the cylinder must have the vertical direction and pass through the axis of the cylinder because the cylinder rolls uniformly. Consequently, the force of rolling friction is given by the relation

$$f_r = P\sin\alpha \approx P\alpha \approx \frac{s}{R}P \tag{75.5}$$

In these experiments it is necessary to check whether the resistance force produced by the viscosity of the air is in fact much smaller than the force of rolling friction.

When a cart is rolled on the ground the forces of rolling friction acting on its wheels obviously depend on the velocity; the experimental data concerning such a dependence are still rather poor. The problem of the computation of the

force of rolling friction for the wheel of an automobile is much more complex. In the first approximation it can be assumed that the moment of this friction force is also constant and that the value of the coefficient s needed for the computation can be taken from the corresponding tables and can be considered dependent solely on the parameters characterizing the tyres of the automobile and the road surface.

The force of rolling friction of a cylinder is much smaller than the force of sliding friction and therefore in modern machines ball and roller bearings are

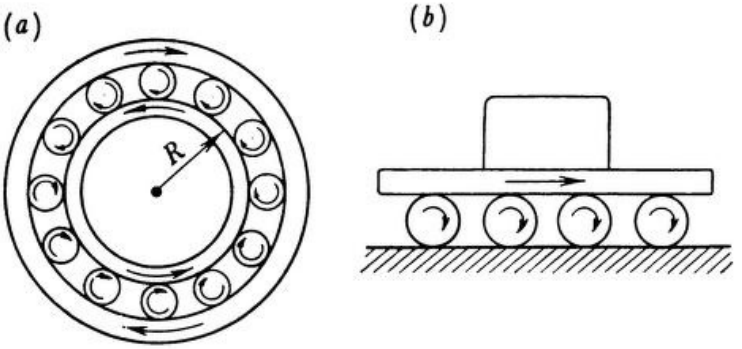

FIG. 212

widely used. As can be seen from Fig. 212, in the case of a plane surface of contact (Fig. 212*b*) the rollers are in a pure rolling motion while in an ordinary ball or roller bearing (Fig. 212*a*) the rollers cannot be in a pure rolling motion and there must appear sliding. However, the smaller the ratio of the radius of the balls (rollers) to the radius R of the inner race of the bearing, the weaker is the sliding. On the other hand, the balls (rollers) cannot have a very small radius because they press in the surface of the races of the bearing.

If the material of a cylinder and of an inclined plane down which the cylinder rolls is sufficiently hard the force of rolling friction is very small. This makes it possible to study the laws of a uniformly accelerated motion in experiments with a ball rolling down an inclined trough or with a cylinder rolling down an inclined plane by analogy with experiments with bodies falling in vacuum. This was very important for Galileo's experiments in which he studied the laws of free fall by observing a ball rolling down an inclined trough.

Let γ be the angle of inclination of the plane to the horizontal direction and let γ be greater than α (see (75.5)). Then the accelerating force is equal to $mg \sin \gamma$. The force of rolling friction is equal to $mg \sin \alpha$. If the air resistance is negligibly small then, by formulas (58.10) and (59.12), the acceleration of the ball is constant and is equal to

$$a = \frac{5}{7} g (\sin \gamma - \sin \alpha) \tag{75.6}$$

Choosing γ in an appropriate manner we can make the acceleration sufficiently small, which makes it possible to observe an accelerated motion with a small velocity; as was shown in Sec. 40, in such a case it is in fact legitimate to neglect air friction. *

* However, in Sec. 40 we dealt with a translatory motion while here we speak of a compound motion of a ball which is a combination of a translatory and a rotational motion. But when the angular velocity and the linear velocity of the ball are small the air resistance is of the same order as in the case of a pure translatory motion.